Zero Modes of Rotationally Symmetric Generalized Vortices and Vortex Scattering

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Abstract Zero modes of rotationally symmetric vortices in a hierarchy of generalized Abelian Higgs models are studied. Under the finite-energy and the smoothness condition, it is shown, that in all models, n self-dual vortices superimposed at the origin have $2n$ modes. The relevance of these modes for vortex scattering is discussed, first in the context of the slow-motion approximation. Then a corresponding Cauchy problem for an all head-on collision of n vortices is formulated. It is shown that the solution of this Cauchy problem has a $\frac{\pi}{n}$ symmetry.

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1. Introduction

Since their discovery [1], vortices in the Abelian Higgs model have attracted much attention. This is mainly due to the fact that the static solutions of this model describe flux tubes in superconductors. As objects in 2-dimensional space, they also provide simple examples of topologically nontrivial structures. Recently , the Abelian Higgs model was generalized, and rotationally symmetric vortices were found in these models [2]. The *generalised* Abelian Higgs models form a hierarchy of 2 dimensional $U(1)$ models lablled by an integer parameter p. The significance of p is that the p-th member of this hierarchy has been derived by subjecting the p-th member of the hierarchy of $4p$ dimensional scale-invariant $SO(4p)$ Yang-Mills models to dimensional descent[3]. The $p = 1$ members of both these hierarchies are the usual Abelian Higgs model and the Yang-mills model in 2 and 4 dimensions respectively.

A mathematically rigorous proof for the existence and uniqueness of the rotationally symmetric self-dual vortices of the hierarchy of generalised Abelian Higgs models was subsequently given in [4]. These rotationally symmetric solutions describe vortices superimposed at the origin, and provide us with objects qualitatively similar to, but quantitatively different from the vortices in the Abelian Higgs model. In the case of the Abelian Higgs model, actually, a $2n$-parameter family of vortices exists [5]. In this paper, we show that, at least near the rotationally symmetric vortices, there is also a $2n$-parameter family of generalized vortices.

In the Abelian Higgs model, the knowledge gained from the study of the zero modes has been used to investigate the scattering of vortices. Vortex scattering has been studied in the context of the slow-motion approximation [6], and with the help of the theory of partial differential equations [7]. Among the most interesting processes studied is $90°$ scattering of 2 vortices in a head-on collision (or its generalization: $\frac{\pi}{n}$ scattering in all head-on collisions of n vortices.) We will show that some of the results, in particular the $\frac{\pi}{n}$ symmetry of the scattering process, generalize to generalized vortices.

The paper is organized as follows. In Sec. 2, we highlight those aspects of the generalized Abelian Higgs models, and of their rotationally symmetric vortices, important to our discussion of the zero modes. In Sec. 3, following steps taken by Weinberg in the case of the Abelian Higgs model [8], we derive the fluctuation equations for rotationally symmetric generalized vortices. We

show that these equations have a $2n$-parameter family of smooth finite-energy solutions. In Sec. 4, the scattering of n generalized vortices shortly before and after they form a rotationally symmetric vortex is discussed, first, briefly, in the context of the slow-motion approximation. Then a corresponding Cauchy problem is formulated. It is shown that its solution has a $\frac{\pi}{n}$ symmetry.

2. The Models and their Radially Symmetric Solutions

The models we study are the generalized Abelian Higgs models in (2+1)-dimensional space-time, given by the Lagrangian densities,

$$\begin{aligned} \mathcal{L}^{(p)} &= (\eta^2 - |\phi|^2)^{2(p-2)}([(\eta^2 - |\phi|^2)F_{\mu\nu} + \imath(p-1)D_{[\mu}\phi^* D_{\nu]}\phi]^2 \\ &\quad +4p(2p-1)(\eta^2 - |\phi|^2)^2|D_\mu\phi|^2 + 2(2p-1)^2(\eta^2 - |\phi|^2)^4), \end{aligned} \tag{1}$$

for any integer $p > 1$. For $p = 1$, (1) reduces to the usual Abelian Higgs model. ϕ is the complex Higgs field, $D_\mu\phi = \partial_\mu\phi + \imath A_\mu\phi$, $\mu = 0, 1, 2$, is the covariant derivative, and the gauge fields $F_{\mu\nu}$ are defined in terms of the real gauge potentials A_μ as $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu = \partial_{[\mu}A_{\nu]}$ $\mu, \nu = 0, 1, 2$. The square brackets mean antisymmetrization in the indices. For any tensor $T_{\mu\nu\ldots}$, $(T_{\mu\nu\ldots})^2$ means $T_{\mu\nu\ldots}T^{\mu\nu\ldots}$. The indices are raised and lowered with the metric tensor $g = diag(-1, +1, +1)$. Our task is to find, and study, smooth finite-energy solutions of the corresponding Euler-Lagrange equations.

As in the case of the Abelian Higgs model, the Euler-Lagrange equations of the generalized Abelian Higgs models are solved by certain first-order equations. For the Abelian Higgs model, these equations were found by Bogomol'nyi [9]. For the Lagrangian (1), the first-order equations arise as follows: In the case $A_0 = 0$, and for time-independent Higgs field and gauge potentials, the Lagragian can be written in the form, $\mathcal{L}^{(p)} = I^2 + J^2 + \epsilon_{ij}\partial^i\Omega^j$, where I^2 and J^2 are the positive definite terms,

$$\begin{aligned} I^2 &= (\eta^2 - |\phi|^2)^{2(p-2)}(((\eta^2 - |\phi|^2)F_{ij} \\ &\quad +\imath(p-1)D_{[i}\phi^* D_{j]}\phi) - (2p-1)\epsilon_{ij}(\eta^2 - |\phi|^2)^2)^2, \end{aligned} \tag{2}$$

$$J^2 = 2p(2p-1)(\eta^2 - |\phi|^2)^{2(p-1)}|D_i\phi - \imath\epsilon_{ij}D^j\phi|^2, \tag{3}$$

and where,

$$
\begin{aligned}
\Omega^j &= 4(2p-1)\eta^{2(2p-1)}A^j \\
&\quad + \sum_{s=1}^{2p-1}\frac{4\imath}{s}(2p-1)^2\eta^{2(2p-1-s)}\begin{pmatrix} 2p-2 \\ s-1 \end{pmatrix}(-|\phi|^2)^{s-1}\phi^* D^j\phi.
\end{aligned} \tag{4}
$$

The special structure of the Lagrangian, $\mathcal{L}^{(p)}$, is no accident, but arises naturally through dimensional reduction of the generalized Yang-Mills model on $R^2 \times S^{4p-2}$ [2]. The divergence, $\epsilon_{ij}\partial^i\Omega^j$, is the dimensionally reduced form of the Chern-Pontryagin density. $I = 0$ and $J = 0$ stem from the self-duality equations of the generalized Yang-Mills theory [3]. Setting I and J equal to zero, means minimizing the energy in a given tological sector, and yields the desired first-order equations. We have simplified the task of finding time-independent solutions with $A_0 = 0$, to solving the following equations:

$$
(\eta^2 - |\phi|^2)F_{ij} + \imath(p-1)D_{[i}\phi^* D_{j]}\phi = (2p-1)\epsilon_{ij}(\eta^2 - |\phi|^2)^2, \tag{5}
$$

$$
D_i\phi = \imath\epsilon_{ij}D^j\phi. \tag{6}
$$

For the rotationally symmetric ansatz,

$$
\phi = \eta g(r)\exp(-\imath n\theta), \tag{7}
$$

$$
A_i = \epsilon_{ij}x^j\frac{a(r)-n}{r^2}, \tag{8}
$$

with integer vortex number n, Eqs (5) and (6) reduce to,

$$
(1-g^2)\frac{da}{dr} = \frac{2}{r}(p-1)a^2g^2 - \eta^2(2p-1)r(1-g^2)^2, \tag{9}
$$

$$
\frac{dg}{dr} = \frac{ag}{r}. \tag{10}
$$

In Ref. 3, for all n, existence and uniqueness of a solution to these equations with the desired asymptotic behaviour was shown. For small r, a goes to n, and g goes like $C_n r^n$, where the C_n are constants which have to be determined numerically. For large r, g goes to 1, and $a/(1-g^2)$ goes like $\sqrt{(2p-1)/(2p)}\eta r$. In the next section, we will study the zero modes of these rotationally symmetric solutions.

3. The Fluctuation Equations and the Zero Modes

We write the fluctuations about the rotationally symmetric solutions in the following form:

$$\delta\phi = \eta g(r) h(r,\theta) \exp(-\imath n\theta), \tag{11}$$

$$\delta A_1 = \frac{1}{r}(-b(r,\theta)\sin\theta + c(r,\theta)\cos\theta), \tag{12}$$

$$\delta A_2 = \frac{1}{r}(b(r,\theta)\cos\theta + c(r,\theta)\sin\theta). \tag{13}$$

Here, b and c are real functions, and $h = h^1 + \imath h^2$ is a complex function. To count the number of modes, we will later fix the gauge by setting h^2 equal to zero. To discuss the smoothness of the modes, we will have to use the gauge freedom and find suitable functions h^2.

Equations (5) and (6), linearized in the functions h, b and c, then yield the fluctuation equations. Equation (6) leads to expressions for the functions b and c in terms of the function h:

$$b = -r\frac{\partial h^1}{\partial r} - \frac{\partial h^2}{\partial \theta}, \tag{14}$$

$$c = \frac{\partial h^1}{\partial \theta} - r\frac{\partial h^2}{\partial r}. \tag{15}$$

Using these expressions, we obtain from eq. (5) the following equation for h:

$$\frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial h^1}{\partial r}) + \frac{1}{r^2}\frac{\partial^2 h^1}{\partial \theta^2} - \frac{4(p-1)ag^2}{r(1-g^2)}\frac{\partial h^1}{\partial r} - g^2(2(2p-1)\eta^2 + \frac{4(p-1)a^2}{r^2(1-g^2)^2})h^1 = 0. \tag{16}$$

Note that h^2 does not occur in Eq. (16). Therefore, the finite-energy solutions of Eq. (16), which lead to smooth fields in a suitable gauge, yield the zero modes. There are no other modes, since we can impose the gauge condition, $h^2 = 0$.

We attempt to solve Eq. (16) in terms of a Fourier series for h^1,

$$h^1(r,\theta) = \sum_{k=0}^{\infty}(h_k^{(1)}(r)\cos k\theta + h_k^{(2)}(r)\sin k\theta). \tag{17}$$

The Fourier coefficient functions, $h_k^{(i)}$, have to satisfy the equation,

$$\frac{d^2h_k^{(i)}}{dr^2}+(\frac{1}{r}-\frac{4(p-1)ag^2}{r(1-g^2)})\frac{dh_k^{(i)}}{dr}-(\frac{k^2}{r^2}+2(2p-1)\eta^2g^2+\frac{4(p-1)a^2g^2}{r^2(1-g^2)^2})h_k^{(i)}=0. \tag{18}$$

We want to find all solutions such that, in a suitable gauge, the rotationally symmetric background fields plus the fluctuations are C^∞ functions on R^2 with finite energy.

For small r, Eq. (18) reduces to, $r^2y''+ry'=k^2y$, where $y=h_k^{(i)}$. The general solution to this equation is, $y=c_1r^k+c_2r^{-k}$ for $k>0$, and $y=c_1+c_2\ln r$ for $k=0$. The uniqueness of radially symmetric vortices rules out nontrivial fluctuations for $k=0$. (We assume here that the uniqueness result, rigorously proven in Ref. 4 for $p=2$, is in fact true for all p.) For large r,

$$y''-(p-1)\sqrt{\frac{8(2p-1)}{p}}\eta y'-\frac{2(2p-1)^2}{p}\eta^2y=0, \tag{19}$$

holds, with general solution,

$$y=c_3\exp(\sqrt{\frac{2}{p}}(2p-1)^{\frac{3}{2}}\eta r)+c_4\exp(-\sqrt{\frac{2(2p-1)}{p}}\eta r). \tag{20}$$

Because of the finite-energy condition, we have to set $c_3=0$. c_4 is an arbitrary coefficient. One can now show that all exponentially decreasing solutions at infinity lead to solutions at the origin with $c_2\neq 0$. Assume the contrary: Then for $c_4>0$ there must be a maximum with positive y. However, Eq. (18) shows that, when the first derivative vanishes, the second derivative is positive. This is a contradiction. (An analogous argument holds when c_4 is negative and y has to attain a minimum.) Hence, acceptable nontrivial behaviour at infinity leads to an r^{-k} term at the origin. For $k>n$ this implies that the energy is infinite.

We are left with the $2n$ modes $h_k^{(i)}$ (with all the other $h_l^{(j)}$, for $j\neq i$ or $l\neq k$, equal to zero) for $0<k\leq n$ and $i=1,2$. Supplemented with suitable functions h^2, these modes lead to the following smooth finite-energy solutions: For the first set of n functions, the fluctuations are of the form,

$$\delta\phi=\eta g(r)h_k^{(1)}(r)\exp(-\imath(n-k)\theta), \tag{21}$$

$$\delta A_1=-(\frac{dh_k^{(1)}}{dr}+\frac{k}{r}h_k^{(1)})\sin((k-1)\theta), \tag{22}$$

$$\delta A_2 = -(\frac{dh_k^{(1)}}{dr} + \frac{k}{r}h_k^{(1)})\cos((k-1)\theta). \tag{23}$$

For the second set of n functions, the fluctuations are of the form,

$$\delta\phi = -\imath\eta g(r)h_k^{(2)}(r)\exp(-\imath(n-k)\theta), \tag{24}$$

$$\delta A_1 = (\frac{dh_k^{(2)}}{dr} + \frac{k}{r}h_k^{(2)})\cos((k-1)\theta), \tag{25}$$

$$\delta A_2 = -(\frac{dh_k^{(2)}}{dr} + \frac{k}{r}h_k^{(2)})\sin((k-1)\theta). \tag{26}$$

4. Symmetries and Vortex Scattering

In the Abelian Higgs model, the zero modes can be used to study vortex scattering in the context of the slow-motion approximation [10]. The idea of this approximation is that for low velocities, at each point in time the fields, A_1, A_2, and ϕ, are given by one of the static solutions; i.e., as an ansatz for these functions one can choose the family of static solutions after making the parameters time dependent. A_0 is then determined from its equation of motion, which is of zero order in t. Finally, with this ansatz the action is minimized.

For our purpose, this idea can be implemented as follows. Near the rotationally symmetric vortices, we expand the fields,

$$\phi^i = \hat{\phi}^i + s^i(t)\delta\phi^i, \quad A^i = \hat{A}^i + s^{2+i}(t)\delta A^i \tag{27}$$

Here, $\hat{\phi} = \hat{\phi}^1 + \imath\hat{\phi}^2$ and $\hat{A}^i$ are the the static solutions (7) and (8). $\delta\phi^i$ and δA^i are the fluctuations (21,22,23) or (24,25,26). The equation, which has to be solved for A_0, is the first (for $\nu = 0$) of the following three equations:

$$\begin{aligned}\partial_\mu((\eta^2 - |\phi|^2)^{2p-3}((\eta^2 - |\phi|^2)F^{\mu\nu} &+ \imath(p-1)D^{[\mu}\phi^* D^{\nu]}\phi)) + (p-1) \\ \times(\eta^2 - |\phi|^2)^{2p-4}(((\eta^2 - |\phi|^2)F^{\mu\nu} &+ \imath(p-1)D^{[\mu}\phi^* D^{\nu]}\phi)(\phi^* D_\mu\phi + \phi D_\mu\phi^*)) \\ -\imath p(2p-1)(\eta^2 - |\phi|^2)^{2p-2} &\times (\phi D^\nu\phi^* - \phi^* D^\nu\phi) = 0. \end{aligned} \tag{28}$$

(The other two equations (for $\nu = 1, 2$) are the second-order equations of motion for A_1 and A_2, which we will need later.)

The dynamics, as discribed by the functions $s^\alpha(t)$, is now given by the Lagrange function,

$$\begin{aligned} L^{(p)} &= \int_{R^2} (\eta^2 - |\phi|^2)^{2(p-2)} (2((\eta^2 - |\phi|^2) F_{0i} + \imath(p-1) D_{[0}\phi^* D_{i]}\phi)^2 \\ &\quad +4p(2p-1)(\eta^2 - |\phi|^2)^2 |D_0\phi|^2)\, d^2x. \end{aligned} \tag{29}$$

The functions s^α are found by solving the Euler-Lagrange equations of this Lagrange function. Obviously, the slow-motion approximation still leaves us with very complicated equations to solve. With our ansatz (27), however, we only attempt to study the neighbourhood of rotationally symmetric vortices; i.e., we can neglegt higher order terms in the functions s^α. Even though we are allowed to linearize in s^α, we were not able to solve Eq. (28) for $\nu = 0$.

In the Abelian Higgs model, the Gauss equation corresponds to Eq. (28) for $\nu = 0$. Following the same steps we have just discussed for our models, in the Abelian Higgs model one finds that $A_0 = 0$, and that the s^α are functions linear in time, in the neighbourhood of the rotationally symmetric vortices. For the modes with real smooth $\delta\phi$ this implies $\frac{\pi}{n}$ scattering, if only times shortly before and shortly after the vortices coincide are considered. We see that the slow-motion approximation yields more results in the Abelian Higgs model. Another difference between the models discussed here and the Abelian Higgs model is that in the Abelian Higgs model the validity of the slow-motion approximation has been proved rigorously [11].

Instead of pursuing the slow-motion approximation any further, we will now formulate a corresponding Cauchy problem and use the symmetries to study the full Euler-Lagrange equations of the Lagrangian (1). We work with the vector, $\psi^T = (A_0, \partial_t A_0, A_1, \partial_t A_1, A_2, \partial_t A_2, \phi, \partial_t \phi)$, and impose the following initial conditions:

$$\psi(0, \vec{x})^T = (\hat{A}_0, 0, \hat{A}_1, \delta A_1, \hat{A}_2, \delta A_2, \hat{\phi}, \delta\phi) \tag{30}$$

Here, $\hat{\phi}$ and $\hat{A}_i$ are the static solutions (7) and (8). $\delta\phi$ and δA_i are the fluctuation (21,22,23) for $k = n$. We concentrate on this type of fluctuations for simplicity, and because they lead to the interesting $\frac{\pi}{n}$ symmetry in the n vortex scattering process. $\hat{A}_0$ is the solution of Eq. (28) for $\nu = 0$ at time $t = 0$.

The equations of motion we consider are the following second-order equations: The second-order equation for A_0 is, $\partial_{tt}A^0 + \partial_i\partial_t A^i = 0$. (This equation

follows from the Lorentz condition $\partial_\mu A^\mu = 0$.) The second-order equations for A_1 and A_2 are given by Eq. (28) for $\nu = 1, 2$. The second-order equation for ϕ is its Euler-Lagrange equation,

$$\begin{aligned}
\imath\partial_\mu((\eta^2 - |\phi|^2)^{2p-4}[(\eta^2 - |\phi|^2)F^{\nu\mu} &+ \imath(p-1)D^{[\nu}\phi^* D^{\mu]}\phi](p-1)D_\nu\phi^* \\
+p(2p-1)(\eta^2 - |\phi|^2)^2 D^\mu\phi^*) &+ \frac{1}{2}(p-2)(\eta^2 - |\phi|^2)^{2p-3} \times \\
([(\eta^2 - |\phi|^2)F_{\mu\nu} + \imath(p-1)D_{[\mu}\phi^* D_{\nu]}\phi]^2 &+ 4p(2p-1)(\eta^2 - |\phi|^2)^2|D_\mu\phi|^2 \\
+2(2p-1)^2(\eta^2 - |\phi|^2)^4)\phi^* &- \frac{1}{2}(\eta^2 - |\phi|^2)^{2(p-2)}(((\eta^2 - |\phi|^2) \\
\times F^{\mu\nu} + \imath(p-1)D^{[\mu}\phi^* D^{\nu]}\phi) &\times (F_{\mu\nu}\phi^* - (p-1)A_{[\mu}D_{\nu]}\phi^*) \\
-2p(2p-1)(\eta^2 - |\phi|^2)|D_\mu\phi|^2\phi^* &+ \imath p(2p-1)(\eta^2 - |\phi|^2)^2 A^\mu D_\mu\phi^* \\
-2(2p-1)^2(\eta^2 - |\phi|^2)^3\phi^*) &= 0. \qquad (31)
\end{aligned}$$

The Lorentz condition and Eq. (28) for $\nu = 0$ are considered as constraints, which, by our choice of initial data, are satisfied at $t = 0$. In the Abelian Higgs model, it has been proved that the analogous Cauchy problem has a unique solution, and that the constraints are propagated [7]. In the following, we will assume that also in the present case a unique solution exists, although a rigorous proof is still missing. (Because of the complexity of the equations, to rigorously prove existence of a unique solution seems a very difficult task.)

For a given solution $\psi(t, \vec{x})$, we define the functions, $\psi_i(t, \vec{x}) = M_i\psi(t, \vec{x}_{(i)})$ for $i = 1, 2$, where $\vec{x}_{(1)} = S\vec{x}$ with

$$S = \begin{pmatrix} \cos\frac{2\pi}{n} & -\sin\frac{2\pi}{n} \\ \sin\frac{2\pi}{n} & \cos\frac{2\pi}{n} \end{pmatrix}, \qquad (32)$$

and where $\vec{x}^T_{(2)} = (x_1, -x_2)^T$. The matrices M_i are defined as,

$$M_1 = \begin{pmatrix} I & 0 & 0 & 0 \\ 0 & A & B & 0 \\ 0 & -B & A & 0 \\ 0 & 0 & 0 & I \end{pmatrix}, I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, A = \cos\frac{2\pi}{n}I, B = \sin\frac{2\pi}{n}I, \qquad (33)$$

and

$$M_2 = \begin{pmatrix} -I & 0 & 0 & 0 \\ 0 & -I & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & C \end{pmatrix}, \quad CV = V^*. \qquad (34)$$

We can now show that, if $\psi(t,\vec{x})$ is a solution of the Cauchy problem, so are $\psi_1(t,\vec{x})$ and $\psi_2(t,\vec{x})$. (To find the symmetries of the initial data $A_0(0,\vec{x})$, we have assumed that Eq. (28) for $\nu = 0$ has a unique smooth solution with asymptotic decay sufficient to satisfy the finite-energy condition.) The uniqueness of the solution of the Cauchy problem now implies that, actually, ψ, ψ_1, and ψ_2 are all the same function. This, in turn, implies that functions like $|\phi|^2$, F_{ij}^2, or the energy density $\mathcal{E}$ are invariant under a $\frac{2\pi}{n}$ rotation, and under a reflection w.r.t. the x_1-axis. This leads to the following conclusion: If by using functions like $|\phi|^2$,F_{ij}^2 or $\mathcal{E}$, there is a way of defining the positions $(x_1^a(t), x_2^a(t)), a = 1, ..., n,$ of exactly n separate vortices, these n positions must lie on n radial lines separated by an angle $\frac{2\pi}{n}$ with equal distance from the origin. (As in Ref. 7, we can use the the minima of $|\Phi|^2$ to define these positions, near the rotationally symmetric vortices.) Furthermore, one of these radial lines must be the positive x_1-axis, or make an angle $\frac{\pi}{n}$ with the positive x_1-axis. Any vortex that does not satisfy these conditions immediately leads to $2n-1$ other vortices, because of the symmetries of our solution. For continuous solutions, these positions will change continuously such that at $t = 0$ the n positions coincide, and after the collision the vortices move again on the radial lines just described. Therefore, they can either go back on the radial lines they came in on, or go back on radial lines shifted by an angle $\frac{\pi}{n}$. We will study a further symmetry to show that the second case is realised.

The last transformation we study is $\vec{x} \to M\vec{x}$, where M is the orthogonal matrix

$$M = \begin{pmatrix} \cos\frac{\pi}{n} & -\sin\frac{\pi}{n} \\ \sin\frac{\pi}{n} & \cos\frac{\pi}{n} \end{pmatrix}. \tag{35}$$

Under this transformation the initial data change as follows:

$$\psi(0, M\vec{x}) = M_3\psi(0,\vec{x}),$$

with

$$M_3 = \begin{pmatrix} -\sigma & 0 & 0 & 0 \\ 0 & C & -D & 0 \\ 0 & D & C & 0 \\ 0 & 0 & 0 & -\sigma \end{pmatrix}\ ,\ \sigma = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\ , C = \cos\frac{\pi}{n}\sigma\ ,\ D = \sin\frac{\pi}{n}\sigma. \tag{36}$$

(We have again assumed the uniqueness of the smooth finite-energy solution, $A_0(0,\vec{x})$, of Eq. (28) for $\nu = 0$.) From the uniqueness of the solution of the Cauchy problem, $\psi(-t, M\vec{x}) = M_3\psi(t,\vec{x})$ follows, and we see that the functions, $|\phi|^2$, F_{ij}^2 and $\mathcal{E}$, are invariant under the transformation $(t,\vec{x}) \to (-t, M\vec{x})$. This establishes $\frac{\pi}{n}$ scattering for n vortices.

5. Conclusions

The study of vortices in a hierarchy of generalized Abelian Higgs models, and the comparison with the Abelian Higgs model was continued. In previous studies we had seen that the rotationally symmetric generalized vortices are qualitatively similar to, but quantitatively different from the vortices. Here similar structures were found in the neighbourhood of the rotationally symmetric vortices. We found that the zero modes of the generalized vortices have the same angular dependence as, but radial behaviour different from that of the zero modes of the vortices in the Abelian Higgs model.

The slow-motion approximation turned out to be of limited value. (Since the time-independent solutions are not known explicitly even in the Abelian Higgs model, the slow-motion approximation is not very successful in this model either.) On the reasonable assumption that a certain Cauchy problem has a unique solution, we were, however, able to study the symmetries of certain solutions. Each solution describes a process where n vortices approach and form one structure, namely a rotationally symmetric vortex. From this structure n vortices emerge. The pattern the vortices create for time t is the same as that for time $-t$, after a $\frac{\pi}{n}$ rotation.

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